TD10 : Maths

$$\left(1-\frac{1}{n^2}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{n^{2k}}\right) = \left(1-\frac{n}{n^2}+\left[\frac{\binom{n}{2}}{n^4}-\frac{\binom{n}{3}}{n^6}\cdots(-1)^n\frac{\binom{n}{n}}{n^{2n}}\right]\right)$$

6. $$n^2\binom{n}{2k} > \binom{n}{2k+1} \quad 1 \leq k \leq E(n/2)$$

$$n^2\left(\frac{\cancel{n!}}{(n-2k)!\,\cancel{2k!}}\right) > \frac{\cancel{n!}}{(n-2k-1)!\,(2k+1)\cancel{2k!}}$$

$$\frac{n^2}{(n-2k)\cancel{(n-2k-1)!}} > \frac{1}{\cancel{(n-2k-1)!}\,(2k+1)} \Rightarrow \frac{n^2}{n-2k} > \frac{1}{2k+1} \text{ quand } 1 \leq k < E(n/2)$$

$$\frac{n^2}{n-2k} > \frac{1}{2k+1}$$

$$\Rightarrow \frac{n^2(2k+1)}{n-2k} > 1$$

$$\left(1-\frac{1}{n^2}\right)^n - \left(1-\frac{1}{n}\right) = \left\{\frac{\binom{n}{2}}{n^4}-\frac{\binom{n}{3}}{n^6}\right\} + \left\{\frac{\binom{n}{4}}{n^8}\ \frac{\binom{5}{n}}{n^{10}}\right\} + \dots + \underline{(-1)^n\binom{n}{n}}$$

$$\sum_{k=2}^{n}\left(-\frac{1}{n^2}\right)\binom{n}{k} - \left(1-\frac{1}{n}\right)$$ Si n pair, somme de nb positif > 0

$n-1$ impair somme de nb positif $> 0 + \frac{1}{n^2} > 0$

Donc u_n est bornée et converge.

Théorème de Leibniz

$(u_n)_{n\geq 0}$ $u_n > 0$, $u_{n+1} < u_n$, $\lim_{n\to+\infty} u_n = 0$.

La suite $S_n = \sum_{k=0}^{n} (-1)^k u_k$ converge.

Suites adjacentes

(1) $u_n < v_n$

(2) $u_n \nearrow$ et $v_n \searrow$

(3) $\lim_{n\to+\infty} u_n - v_n = 0$.

2.27. (u_n) $\searrow$, $\lim u_n = 0$

$S_n = \sum_{k=0}^{n} (-1)^k u_n$ converge.

S_{2n} et S_{2n+1} sont adjacentes.

$S_{2(n+1)} \leq S_{2n} \iff -u_{2n+1} + u_{2n+2} \leq 0$ donc vrai

$u_n \searrow$ donc $u_{2n+1} \geq u_{2n+2}$

$S_{2n+1} \leq S_{2n+3} \iff S_{2n+1} - S_{2n+3} \leq 0$

$\iff \cancel{u_0} - \cancel{u_1} + \cancel{u_2} + \dots - \cancel{u_{2n+1}} - (\cancel{u_0} - \cancel{u_1} + \dots - \cancel{u_{2n+1}} + u_{2n+2} - u_{2n+3}) \leq 0$

$\iff u_{2n+2} - u_{2n+3} \leq 0$.

$u_{2n+2} \leq u_{2n+3}$

$\lim\limits_{n \to +\infty} S_{2n} - S_{2n+1} = -U_{2n+1} = 0$.

2.28.

$f_n(x)\ x^n + 9x^2 - 4 \qquad f_n : \mathbb{R}_+ \longmapsto \mathbb{R}, \quad n \in \mathbb{N}^*$.

$f_n(x)$ admet une unique solution $u_n \in \mathbb{R}_+^*$.

$f_1(x) = x + 9x^2 - 4$

$f_2(x) = x^2 + 9x^2 - 4$

$f_3(x) = x^3 + 9x - 4$.

u_1 : sol° de $f_1(x) = 0$. $\Rightarrow 9x^2 + x - 4$. $\Delta = 1 + 144 = 145$.

$$x_1 = \frac{-1-\sqrt{145}}{18} < 0 \quad x_2 \frac{-1+\sqrt{145}}{18} > 0$$

$$u_1 = \frac{-1+\sqrt{145}}{18},$$

u_2 sol° de $f_2(x) = 0 \Rightarrow 10x^2 - 4 = 0 \Leftrightarrow x^2 = \frac{2}{5} \Rightarrow x = \sqrt{\frac{2}{5}}$.

$$u_2 = \sqrt{\frac{2}{5}}.$$

$f_n(x) = 0$ possède une unique solut° $u_n = 0$.

$x^n + 9x^2 - 4 = 0$.

$f_n'(x) = nx^{n-1} + 18x > 0 \quad s^t \nearrow$,

Donc $f_n'(x)$ possède tout au plus une racine positive

$\rightarrow$ TVI. ccl.

Mq $0 < u_n < \frac{2}{3}$.

$f_n(0) < f_n(u_n) < f_n\left(\frac{2}{3}\right)$ (croissance de f_n)

$-4 < 0 \leq \left(\frac{2}{3}\right)^n + \cancel{9} \times \frac{4}{\cancel{9}} \cancel{-4}$ C'est vrai donc $u_n \in]0; \frac{2}{3}[$.

Mq. $\forall x \in]0;1[$, $f_{n+1}(x) < f_n(x)$.

$x^{n+1} \cancel{+ 9x^2} \cancel{-4} < x^n \cancel{+ 9x^2} \cancel{-4}$

$x^{n+1} < x^n$. vrai car $0 < x < 1$.

$x^n x - x^n < 0$

$x^n (x-1) < 0$.

>0 <0 donc ok.

On a : $f_{n+1}(u_{n+1}) < f_n(u_{n+1}) \Rightarrow 0 < f_n(u_{n+1})$.

Mq (u_n) monotone.

On a $f_n(u_{n+1}) > 0$.

$f_n(u_{n+1}) > f_n(u_n) \Leftrightarrow u_{n+1} > u_n$

Donc $(u_n) \nearrow$

Elle est croissante et majorée donc elle converge vers $l \leq \frac{2}{3}$.

(u_n^n) $f_n(u_n) = u_n^n + 9u_n^2 - 4 =$

$u_n^n = -9u_n^2 + 4$.

0 $4 \sim 9u_n^2 \Rightarrow u_n \sim \frac{2}{3}$.

A faire : 2.29, 2.31, 2.32

Nombres complexes

$$2e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

$|1| = 1$
$\arg(1) = 0$

$|2| = 2$
$\arg(2) = \pi$

$|i| = 1$
$\arg(i) = \frac{\pi}{2}$.

$|1+i\sqrt{3}| = 2$
$\arg(1+i\sqrt{3}) = \frac{\pi}{3}$.

$|1+i| = \sqrt{2}$

$\cos(\theta) = \frac{1}{\sqrt{2}}$

$\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \arg(1+i) = \frac{\pi}{4}$.

$$\frac{(1+i\sqrt{3})^6}{(1+i)^4} = \frac{2^6 e^{i\frac{\pi}{3}\times 6}}{\sqrt{2}^4 \cancel{e^{i\frac{\pi}{4}\times 4}}} = 2^4.$$