TD21 : Maths.

Rappel : déf de limite.

$f :]a;b[\longmapsto \mathbb{R}$; $x_0 \in]a;b[$

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ tq } |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

Exercice 6.5. Calculer, lorsqu'elles existent, les limites suivantes :

(1) $\lim_{x\to 1} \frac{x^4-1}{x-1}$, (2) $\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$, (3) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{2x}$,

(4) $\lim_{x\to 0} \frac{x^2+|x|}{x}$, (5) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x^3-1}$, (6) $\lim_{x\to+\infty} (\sqrt{x+1}-\sqrt{x})$,

(7) $\lim_{x\to-\infty} \frac{x^2+|x|}{x}$, (8) $\lim_{x\to 0} \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+9}}$, (9) $\lim_{x\to+\infty} \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+9}}$.

1. $$\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to 1} \frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{x-1}} = (1^2+1)(1+1) = 4.$$

2. $$\lim_{x \to 2} \frac{\cancel{x^2}\left(1-\frac{4}{x^2}\right)}{\cancel{x^2}\left(1-\frac{1}{x}-\frac{1}{x^2}\right)} = \frac{0}{1-\frac{1}{2}-\frac{1}{4}} = 0.$$

$\lim_{x \to 0} \frac{x}{|x|}$ existe ? $\frac{x}{|x|} = \begin{cases} 1 \text{ si } x > 0 \\ -1 \text{ si } x < 0 \end{cases}$.

Donc $\lim_{x \to 0^-} \frac{x}{|x|} = -1 \neq \lim_{x \to 0^+} \frac{x}{|x|} = 1$ Donc la limite n'existe pas.

6. $$\lim_{x \to +\infty} \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \frac{\cancel{x}+1-\cancel{x}}{\sqrt{x+1}+\sqrt{x}} = 0.$$

Exercice 6.1. (1) Montrer que pour tout $0 < \varepsilon < 1$ et pour $x \in \mathbb{R}$, si $|x-1| < \frac{\varepsilon}{4}$, alors $|x^2+x-2| < \varepsilon$.

(2) En déduire (en utilisant la définition d'une limite) les valeurs de :

$$\lim_{x\to 1} (x^2+x-1) \quad \text{et} \quad \lim_{x\to 1} (x^2+x-2)\cos x.$$

$\lim_{x \to 1} x^2+x-2 = 0 \iff \forall \varepsilon > 0, \exists \delta > \text{ tq } |x-1| < \delta \Rightarrow |x^2+x-1| < \varepsilon$. $\delta = \frac{\varepsilon}{4}$

$x^2+x-2 = (x-1)(x+2)$ ✓

si $|x-1| < \frac{\varepsilon}{4}$, $|x-1||x+2| < \frac{\varepsilon}{4}|x+2| \Rightarrow \frac{\varepsilon}{4} \cdot \cancel{4} = \varepsilon$.

$x < 2$ si $\varepsilon > 4$.

2. $\lim_{x \to 1} x^2 - x + 1 = 1$.

$\forall \varepsilon > 0, \exists \delta > 0, |x-1| < \delta \Rightarrow |x^2 - x - 2| < \varepsilon$.

$\Rightarrow$ On prend $\delta = \frac{\varepsilon}{4}$.

Décomposer $P(X) = X^4 + 5X^3 + 9X^2 + 7X + 2$.

$P'(X) = 4X^3 + 15X^2 + 18X + 7$

-1 est racine triple de P.

$P(X) = (X+1)^3 Q(X)$.

$P(X) = (X+1)^3 (X+2)$,

$$\begin{array}{l|l}
\cancel{X^4} + 5X^3 + 9X^2 + 7X + 2 & X^3 + 3X^2 + 3X + 1 \\
\hline
- \cancel{X^4} - 3X^3 - 3X^2 - X & X + 2. \\
\quad 2X^3 - 6X^2 - 6X + 2 & \\
\qquad\qquad 0 &
\end{array}$$

Décomposer $\frac{4}{(X^2+1)(X-1)^3} = F(X)$.

$\deg(4) < \deg((X^2+1)(X-1)^3)$ Donc.

$F(X) = \frac{aX+b}{X^2+1} + \frac{c}{X-1} + \frac{d}{(X-1)^2} + \frac{e}{(X-1)^3}$

$$\frac{(aX+b)(X-1)^3 + c(X^2+1)(X-1)^2 + d(X^2+1)(X-1) + e(X^2+1)}{(X^2+1) \qquad (X-1)^3}.$$

$= (aX+b)(X^2-2X+1)(X-1) + c(X^2+1)(X^2-2X+1) + d(X^3-X^2+X-1) + e(X^2+1)$

$= (aX+b)(X^3-3X^2+3X-1) + c(X^4-2X^3+X^2+X^2-2X+1) +$

$$= aX^4 - 3aX^3 + 3aX^2 - aX + bX^3 - 3bX^2 + 3bX - b + cX^4 - c2X^3 + cX^2 - 2cX + c$$
$$+ dX^3 - dX^2 + dX - d + eX^2 + e.$$

$$X^4(a+c) + X^3(-3a + b + 2c + d) + X^2(3a - 3b + c - d + e)$$
$$+ X(-a + 3b - 2c + d) + (-b + c - d + e)$$

$a + c = 0.$ $\Rightarrow a = -c$

$-3a + b + 2c + d = 0$ $\quad 5c + b + d = 0. \Rightarrow 5c = -b - d.$

$3a - 3b + c - d + e = 0$ $\quad -2c - 3b - d + e = 0,$

$-a + 3b - 2c + d = 0$ $\quad -c + 3b + d = 0.$

$-b + c - d + e = 4.$ $\quad -b + c + d + e = 4. \Rightarrow 5c + c + e = 4.$

$\Rightarrow 6c + e = 4.$

$e = 4 - 6c.$

$a = -c$

$5c = -b - d \Rightarrow b = -5c - d.$

$-2c - 3b - d + 4 - 6c = 0 \Rightarrow -8c - 3b - d = -4.$

$-c + 3b + d = 0 \Rightarrow +c = +3b + d. \Rightarrow -8c - c = -4.$

$e = 4 - 6c.$ $\quad c \Rightarrow \frac{4}{9}. \quad a = -\frac{4}{9}.$

$e = 4 - 6 \times \frac{4}{9} = \frac{36}{9} - \frac{24}{9} = \frac{12}{9} = \frac{4}{3}.$

$-b - d = \frac{5 \times 4}{9} = \frac{20}{9}. \vdash b = -\frac{20}{9} - d$

$-\frac{4}{9} + \cancel{3} \times \frac{20}{\cancel{3} \times 3} - 2d. = 0.$

$\Rightarrow -\frac{4}{9} - \frac{60}{9} = 2d.$

$= -\frac{64}{18} = \frac{32}{9} = d.$ $\quad b = -5 \times \frac{4}{9} - \frac{32}{9} = -\frac{20}{9} - \frac{32}{9} = -\frac{52}{9}.$

Exercice 6.2. Déterminer en utilisant la définition de la limite

(a) $\lim_{x\to 2}\frac{1}{x+1}$ (b) $\lim_{x\to 1^+}\frac{1}{x-1}$ (c) $\lim_{x\to 1^-}\frac{1}{x-1}$ (d) $\lim_{x\to 2}\sqrt{x+2}$.

a. $\lim_{x\to 2}\frac{1}{x+1}=\frac{1}{3}$

$\Longleftrightarrow$

$\forall\varepsilon>0,\ \exists\delta>0,\ |x-2|<\delta \Rightarrow \left|\frac{1}{x+1}-\frac{1}{3}\right|<\varepsilon.$

$\left|\frac{(x-2)}{-3x-3}\right|<\varepsilon$

$|x-2|<\delta \Rightarrow \frac{|x-2|}{|-3x-3|}<\frac{\cancel{|-3x-3|}\varepsilon}{\cancel{|-3x-3|}}$ ok.

Si $\delta\leqslant 1$ $\quad -1<x-2<1$

$\varepsilon=\frac{\delta}{|-3x-3|} \Rightarrow \delta=|-3x-3|\varepsilon.$

$z^4=\frac{1-i}{1+i\sqrt{3}}.=\frac{(1-i)(1-i\sqrt{3})}{2}=\frac{(1-i\sqrt{3}-i+\sqrt{3}}{2}=\frac{1+\sqrt{3}}{2}+i\frac{-1+\sqrt{3}}{2}.$

$|z^4|=\frac{|1-i|}{|1+\sqrt{3}|}.=\frac{\sqrt{2}}{2}$ $\qquad (1+\sqrt{3})^2=1+2\sqrt{3}+3.$

$\sqrt{2}(\cos\theta+i\sin\theta) \Rightarrow \cos\theta'=\frac{\sqrt{2}}{2};\ \sin\theta'=-\frac{\sqrt{2}}{2}.\ \theta'=-\frac{\pi}{4}+2k\pi,$

$2(\cos\alpha+i\sin\alpha) \Rightarrow \cos\alpha=\frac{1}{2};\ \sin\alpha=\frac{\sqrt{3}}{2}.$

$\alpha=\frac{\pi}{3}+2k'\pi.$

$\arg(z^4)=4\arg(z)=-\frac{\pi}{4}+2k\pi-\frac{\pi}{3}-2k'\pi,$

$=\frac{-3\pi-4\pi}{12}+2\pi(k+k')$

$=\boxed{-\frac{7\pi}{12}}+2(k+k')\pi.$

$$z^4 = \frac{\sqrt{2}}{2} e^{-i\frac{7\pi}{12}}.$$

$$z \begin{cases} |z| = \sqrt[4]{\frac{\sqrt{2}}{2}} = \sqrt{\frac{1}{\sqrt{2}}} \\ \arg(z) = \frac{\arg(z^4)}{4} = \frac{-\frac{7\pi}{12} + 2(K+K')\pi}{4} = -\frac{7\pi}{48} + \frac{K\pi}{2}. \quad 0 \leq K < 4. \end{cases}$$

Exercice 2.15. Montrer, en utilisant la définition de la convergence d'une suite, que la suite $(u_n)_{n\geq 0}$, définie par $u_n = \frac{n+1}{2n+1}$, converge vers $\frac{1}{2}$.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n > N \Rightarrow \left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \varepsilon.$

$$\left| \frac{\cancel{2n} + 1 - \cancel{2n}}{4n+2} \right| < \varepsilon . \Leftrightarrow \left| \frac{1}{4n+2} \right| < \varepsilon$$

$$-\frac{1}{\varepsilon} > 4n+2 > \frac{1}{\varepsilon}$$

$$-\frac{1}{\varepsilon} - 2 > 4n > \frac{1}{\varepsilon} - 2$$

$$\Rightarrow -\frac{1}{4\varepsilon} - \frac{1}{2} > n > \frac{1}{4\varepsilon} - \frac{1}{2}$$

$\Rightarrow N \geq \frac{1}{4\varepsilon} + \frac{1}{2}$ Si $\varepsilon = \frac{1}{1000}$, $N = \frac{1}{4} \cdot 1000 - \frac{1}{2} = 250 - \frac{1}{2} = 250.$