TD 2d : Maths.

Exercice 4.39. Soit $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ l'application linéaire définie sur la base canonique $\mathcal{B} = \{1, X, X^2\}$ par

$$f(1) = 2 - 2X + X^2, \quad f(X) = -1 + X + X^2, \quad \text{et} \quad f(X^2) = -2X + 3X^2.$$

(1) Donner la matrice A de f dans la base $\mathcal{B}$.

(2) On considère $P_1(X) = 1 + X - X^2$, $P_2(X) = 1 - X^2$ et $P_3(X) = 1 - X$. Montrer que la famille $\mathcal{B}' = \{P_1, P_2, P_3\}$ est une base de $\mathbb{R}_2[X]$.

(3) Donner la matrice P de passage de la base $\mathcal{B}$ vers la base $\mathcal{B}'$, c'est-à-dire $P = \mathrm{Mat}(Id, \mathcal{B}', \mathcal{B})$.

(4) Calculer P^{-1}.

(5) Donner la matrice D de f dans la base $\mathcal{B}'$.

1) $A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -2 \\ 1 & 1 & 3 \end{pmatrix}$

$P_1 - P_2 + P_3 = e_1$

$P_1 + P_2 + P_3 - 3(P_1 - P_2 + P_3) = -P_1 + 2P_2 - P_3 = e_2$

$-P_1 + 3P_2 - 2P_3 = e_2.$

$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ $\quad P_{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$P_1 - P_2 + P_3 =$

$+P_1 - 2P_2 + P_3 = \cancel{-1} \cancel{+X} - X^2 \cancel{+2} + 2X^2 \cancel{-1} \cancel{+X} = X^2$

$P_1 - P_2 = \cancel{1} + X \cancel{-X^2} \cancel{-1} \cancel{+X^2} = e_2.$

Exercice 5.1. Calculer les primitives suivantes :

$$\int \frac{1}{(x+1)^3}\,dx, \quad \int \frac{x}{(x+1)^3}\,dx\,, \quad \int x^2(x^3+1)^4\,dx\,, \quad \int \frac{x^2}{x^2+1}\,dx\,,$$

$$\int \sqrt{1-x}\,dx\,, \quad \int \frac{dx}{\sqrt{x+1}}\,, \quad \int \frac{6x+3}{x^2+x+1}\,dx\,, \quad \int (3x^2+2x)e^{2x^3+2x^2+1}\,dx$$

① $\displaystyle\int \frac{1}{(x+1)^3}\,dx = \int (x+1)^{-3}\,dx$. On pose $u = x-1$

$\dfrac{du}{dx} = 1 \Rightarrow du = dx$

$\displaystyle\int u^{-3}\,du = \frac{u^{-2}}{-2} + c = -\frac{1}{2(x-1)^2} + C.$

② $\displaystyle\int \frac{x}{(x+1)^3} = \int \frac{1}{(x+1)^2}\,dx - \int \frac{1}{(x+1)^3}\,dx = \frac{1}{-2+1}(x+1)^{-1} + \frac{1}{2(x-1)^2} + C.$

③. $\int x^2(x^3+1)^4 dx$ On pose $u = x^3+1$. $\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 dx$

$$\frac{1}{3}\int u^4 du = \frac{1}{3}\ \frac{1}{5}\ u^5 = \frac{1}{15}\left(x^3+1\right)^5 + C.$$

④. $$\int \frac{x^2}{x^2+1} = \int \frac{x^2+1-1}{x^2+1} = \int dx - \int \frac{1}{x^2+1} = x - \arctan(x) + C$$

⑤ $$\int \sqrt{1-x}\, dx \Rightarrow \int (1-x)^{\frac{1}{2}} dx = \frac{1}{\frac{3}{2}} (1-x)^{\frac{3}{2}} + C = \frac{2}{\sqrt{(1-x)^3}} + C.$$

⑥. $$\int \frac{dx}{\sqrt{x+1}} = -2\int \frac{dx}{2\sqrt{x+1}} = -2\sqrt{x+1}$$

⑦. $$\int \frac{6x+3}{x^2+x+1} dx = 3\int \frac{2x+1}{x^2+x+1} dx$$

$u = x^2+x+1$

$\frac{du}{dx} = 2x+1.$ $du = 2x+1\, dx$

$$= 3\int \frac{1}{u} du = 3\ln(x^2+x+1) + C.$$

⑧. $$\int (3x^2+2x)\, e^{2x^3+2x^2+1} dx$$

$$u = 2x^3+2x^2+1 \qquad du = 6x^2+4x\, dx$$

$$= \frac{1}{2}\int e^u du = \frac{1}{2}\, e^{2x^3+2x^2+1} + C.$$

Exercice 5.2. A l'aide d'intégrations par parties, calculer les primitives suivantes :

1 $\int x\, e^x\, dx$, 2 $\int x^2\, e^{-x}\, dx$, 3 $\int (x^2 - x)\, e^{2x}\, dx$,

4 $\int \ln x dx$, 5 $\int \ln(1+x^2)\, dx$, 6 $\int x \ln^2 x\, dx$,

7 $\int x \cos^2 x\, dx$, 8 $\int (x^2 - 2x)\cos(2x)\, dx$, 9 $\int (x^2 + x)\sin x\, dx$,

10 $\int x \arctan x\, dx$, 11 $\int \sqrt{1-x^2}\, dx$, 12 $\int \arcsin x\, dx$.

① $\int x e^x dx$. $u = x$ $u' = 1.$

$v' = e^x$ $v = e^x$

$$\int x e^x = x e^x - \int e^x = e^x(x-1).$$

② $\int x^2 e^{-x}$ $u = x^2$ $u' = 2x$

$v = -e^{-x}$ $v' = e^{-x}$

$u = 2x$ $u' = 2$

$v = -e^{-x}$ $v' = e^{-x}$

$$= -x^2 e^{-x} + \int 2x e^{-x} dx$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2\int e^{-x} = -x^2 e^{-x} - 2x e^{-x} + 2e^{x}$$

$$= e^{-x}(-x^2 - 2x + 2).$$

③. $\int (x^2 - x) e^{2x} dx$ $u = x^2 - x$ $u' = 2x - 1$

$v = \frac{1}{2} e^{2x}$ $v' = e^{2x}$

$$= \frac{1}{2}(x^2 - x) e^{2x} - \int (2x-1) e^{2x}$$

$u = 2x - 1$ $u' = 2$

$v = \frac{1}{2} e^{2x}$ $v' = e^{2x}$

$$= \frac{1}{2}(x^2 - x) e^{2x} - \frac{1}{2}\left((2x-1)e^{2x} + 2\int e^{2x}\right) = e^{2x}$$

$$= \frac{1}{2} e^{2x}(x^2 - 3x + 2).$$