TD29: Maths.

Exercice 5.2. A l'aide d'intégrations par parties, calculer les primitives suivantes :

$$\int x\, e^x\, dx\,, \qquad \int x^2\, e^{-x}\, dx\,, \qquad \int (x^2 - x)\, e^{2x}\, dx\,,$$

$$4\ \int \ln x dx\,, \qquad 5\ \int \ln(1+x^2)\, dx\,, \qquad 6\ \int x \ln^2 x\, dx\,,$$

$$7\ \int x \cos^2 x\, dx, \qquad 8\ \int (x^2 - 2x)\cos(2x)\, dx\,, \qquad \int (x^2 + x)\sin x\, dx\,,$$

$$\int x \arctan x\, dx\,, \qquad 11\ \int \sqrt{1-x^2}\, dx\,, \qquad \int \arcsin x\, dx\,.$$

(4) $\int \ln(x)\, dx = x\ln(x) - \int dx = x\ln(x) - x.$

(5) $\int \ln(1+x^2)dx = x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}dx$ $\qquad u = \ln(1+x^2) \quad u' = \frac{2x}{1+x^2}$

$v = x \qquad v' = 1$

$\int \frac{x^2}{1+x^2}dx = \int dx + \arctan(x) \qquad \Rightarrow x\ln(1+x^2) - 2x + 2\arctan(x).$

(6). $\int x \ln^2 x\, dx$ $\qquad u = \ln^2(x) \quad u' = \frac{2}{x}\ln(x)$

$= \frac{1}{2}x^2\ln^2 x - \int x\ln(x)\, dx$ $\qquad v = \frac{1}{2}x^2 \quad v' = x$

$u = \ln(x) \quad u' = \frac{1}{x}$

$v = \frac{1}{2}x^2 \quad v' = x$

$= \frac{1}{2}x^2\ln^2 x - x\ln x + \frac{1}{2}\left(\frac{1}{2}x^2\right) = \frac{1}{2}x^2\ln^2(x) - \frac{x^2}{2}\ln(x) + \frac{1}{4}x^2.$

(7) $\int x\cos^2(x)\, dx$. $\qquad u = x \quad u' = 1$

$v = \frac{1}{2}(1+\cos(2x)) \quad v' = \frac{1}{4} + \sin(2x).$

(8) $\int (x^2 - 2x)\cos(2x)\, dx$ $\qquad u = x^2 - 2x \quad u' = 2x - 2$

$v = \frac{1}{2}\sin(2x) \quad v' = \cos(2x)$

$= \frac{1}{2}(x^2 - 2x)\sin(2x) - \int (x-1)\sin(2x)dx$ $\qquad u = (x-1) \quad u' = 1$

$= \frac{1}{2}(x^2-2x)\sin(2x) + \frac{1}{2}(x-1)\cos(2x) - \int \sin(2x)\, dx$ $\qquad v = -\frac{1}{2}\cos(2x) \quad v' = \sin(2x)$

$$= \frac{1}{2}\left(\left((x^2-2x)\sin(2x) + (x-1)\cos(2x) - \cos(2x)\right)\right).$$

$$= \frac{1}{2}\left(\sin(x)(x^2-2x) + (x-2)\cos(2x)\right)$$

⑨ $\int (x^2-x)\sin(x)\,dx$ $\quad u = x^2 - x \quad u' = 2x-1$

$v = -\cos(x) \quad v' = \sin(x).$

$$= -(x^2-x)\cos(x) + \int (2x+1)\cos(x)\,dx$$

$u = 2x+1 \quad u' = 2$

$v = \sin(x) \quad v' = \cos(x)$

$$= -(x^2-x)\cos(x) + (2x+1)\sin(x) + 2\cos(x)$$

10. $\int x\arctan(x)\,dx$ $\quad u = \arctan(x) \quad u' = \frac{1}{1+x^2}$

$$= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2} \qquad v = \frac{1}{2}x^2 \quad v' = x.$$

$$= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\left(\int dx - \int \frac{1}{1+x^2}\,dx\right)$$

$$= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \arctan(x)$$

$$\arctan(x)\left(\frac{1}{2}x^2+1\right) - \frac{1}{2}x.$$

$$\int \sqrt{1-x^2}\,dx =$$

⑪ $I = \int \sqrt{1-x^2}\,dx =$ $\quad u = \sqrt{1-x^2} \quad u' = -\frac{x}{\sqrt{1-x^2}}$

$v = x \quad v' = 1.$

$$x\sqrt{1-x^2} + \int \frac{x}{\sqrt{1-x^2}} \Rightarrow x\sqrt{1-x} - \int \sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x} + \int \frac{dx}{\sqrt{1-x^2}}$$

$$I = \frac{x+\sqrt{1-x}}{2} + \frac{\arcsin(x)}{2}.$$

$$\int \arcsin(x)\,dx \qquad u = \arcsin(x) \quad u' = \frac{1}{\sqrt{1-x^2}}$$

$$v = x \qquad v' = 1$$

$$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2$

$$du = -2x \Rightarrow x\arcsin(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}} = x\arcsin(x) + \sqrt{1-x^2}.$$

Exercice 5.3. Calculer, à l'aide de changements de variable, les primitives suivantes :

1 $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$, 2 $\displaystyle\int \sqrt{1-x^2}\,x\,dx$, 3 $\displaystyle\int \sqrt{1-x^2}\,dx$, 4 $\displaystyle\int \frac{\ln x}{x}\,dx$,

5 $\displaystyle\int \frac{dx}{x\ln x}$, $\displaystyle\int \tan x\,dx$, $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$, $\displaystyle\int \sin x\,\cos^2 x\,dx$.

$\displaystyle\int \frac{1}{2x\ln(x)+x}\,dx$ $\displaystyle\int \frac{6x}{\sqrt{1-9x^4}}\,dx$

1 $u = x^2+1 \quad du = 2x$

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{x^2+1}.$$

2. $u = 1-x^2 \quad du = -2x$

$$-\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}\, u^{3/2}$$

$$= -\frac{1}{3}(1-x^2)^{3/2}.$$

3 $\displaystyle\int \sqrt{1-x^2}\,dx \qquad x = \sin(u) \quad dx = \cos(u)\,du \sim$

4 $u = \ln(x) \quad du = \frac{1}{x}\,dx$

$$\int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\ln^2(x).$$

$u = \ln(x) \quad du = \frac{1}{x}dx$

5. $$\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}\,du = \ln(u) = \ln(\ln(x)).$$

6. $$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx \qquad u = \cos(x) \quad du = -\sin(x)$$

$$= -\int \frac{du}{u} = \ln(u) = -\ln(\cos(x)).$$