TD 30 : Maths.

Exercice 5.3. Calculer, à l'aide de changements de variable, les primitives suivantes :

$$\int \frac{x}{\sqrt{x^2+1}}\,dx\,, \quad \int \sqrt{1-x^2}\,x\,dx\,, \quad \int \sqrt{1-x^2}\,dx\,, \quad \int \frac{\ln x}{x}\,dx\,,$$

$$\int \frac{dx}{x\ln x}\,, \quad \int \tan x\,dx\,, \quad \int \frac{\sin(x)}{1+\cos^2(x)}\,dx\,, \quad 8 \int \sin x\,\cos^2 x\,dx\,.$$

$$\int \frac{1}{2x\ln(x)+x}\,dx \quad \int \frac{6x}{\sqrt{1-9x^4}}\,dx$$

8 $u = \cos(x)$

$du = -\sin(x)$ $\Rightarrow -\int u^2 du = -\frac{1}{3}u^3 = -\frac{1}{3}\cos^3(x) + c.$

9 $\int \frac{1}{2x\ln(x)+x}\,dx = \int \frac{1}{x}\,\frac{1}{2\ln(x)+1}$

$u = 2\ln(x)+1 \qquad du = \frac{2}{x}\,dx$

$\Rightarrow \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(2\ln(x)+1) + c$

10 $\int \frac{6x}{\sqrt{1-9x^4}}\,dx = \int \frac{6x}{\sqrt{1-(3x^2)^2}}\,dx \qquad u = 3x^2 \quad du = 6x\,dx$

$\Rightarrow \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + c = \arcsin(3x^2) + c.$

Exercice 5.4. Calculer les intégrales suivantes :

$$\int_1^2 x\ln x\,dx\,, \quad \int_1^2 \ln^2 x\,dx\,, \quad \int_1^4 \frac{\ln x}{\sqrt{x}}\,dx\,, \quad \int_0^1 x^2 e^x\,dx\,,$$

$$\int_0^{2\pi} \cos^2 x\,dx\,, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}\,, \quad \int_0^{\pi} x\cos x\,dx\,, \quad \int_0^{\frac{\pi}{2}} x\sin^2 x\,dx\,.$$

$u = \ln(x) \qquad u' = \frac{1}{x}$

1 $\int_1^2 x\ln(x)\,dx = \left[\frac{1}{2}x^2\ln(x)\right]_1^2 - \frac{1}{2}\int_1^2 x\,dx \qquad v = \frac{1}{2}x^2 \qquad v' = x$

$\Rightarrow \frac{1}{2}\left[x^2\ln(x)\right]_1^2 - \frac{1}{2}\times\frac{1}{2}\left[x^2\right]_1^2 = \frac{1}{2}(4\ln(2)) - \frac{1}{4}(4-1)$

$= 2\ln(2) - \frac{3}{4}.$

$u = \ln^2(x) \qquad u' = \frac{2}{x}\ln(x)$

$v = x \qquad v' = 1 \qquad$ 2

② $\int_1^2 \ln^2(x)\,dx = [x\ln^2(x)]_1^2 - 2\int_1^2 \ln(x)\,dx$

$= 2\ln^2(2) - 2\,[x\ln(x) - x]_1^2 = 2\ln^2(2) - 2[(2\ln(2) - 2) - 1]$

$= 2\ln^2(x) - 4\ln 2 + 1$

③ $\int_1^4 \frac{\ln(x)}{\sqrt{x}}\,dx \qquad u = \sqrt{x} \quad du = \frac{1}{2u}\,dx \Rightarrow \frac{1}{2u}\,dx$

$\int_1^2 \frac{\ln(u^2)}{\cancel{u}} 2u\,du = \int_1^2 \frac{2\ln(u)}{\cancel{u}} 2u\,du = 4\int_1^2 \ln(u)\,du = 4\,[u\ln(u) - u]_1^2$

$= 4[(2\ln(2) - 2) - 1)$

$= 8\ln(2) - 8 + 4 = 8\ln(2) - 4.$

④ $\int_0^1 x^2 e^{2x}\,dx = [x^2 e^{2x}]_0^1 - 2\int_0^1 x\,e^{2x}$

$\dots = e - 1.$

⑤ $\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \frac{1+\cos(2x)}{2}\,dx \qquad u = 2x \quad du = 2dx$

$= \int_0^{\pi} 1 + \cos(u)\,du = [x + \sin(u)]_0^{\pi} = \pi + 0 + 1 = \pi + 1.$

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⑥ $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = 2\,[\arcsin(x)]_0^{1/2} = 2 \times \frac{\pi}{6} = \frac{\pi}{3}.$

⑦ $\int_0^{\pi} x\cos(x)\,dx = [x\sin(x)]_0^{\pi} - \int \sin(x)\,dx \qquad u = x \quad u' = 1 \quad v = \sin(x) \quad v' = \cos$

$= 0 + [+\cos(x)]_0^{\pi} = -1 - 1 = -2.$

(8) $\int_0^{\pi/2} x\sin^2(x)\,dx = \int_0^{\pi/2} \frac{x}{2}(1-\cos(2x))\,dx$

$u = x \quad u' = 1$

$v = \frac{1}{2}\sin(2x) \quad v' = \cos(2x)$

$= \cancel{\frac{1}{2}\left[x\sin(2x)\right]_0^{\pi/2}} + \frac{1}{2}\int_0^{\pi/4} \sin(2x)\,dx$

$= \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2} = -\frac{1}{4}[-1-1] = -\frac{1}{2}.$

Exercice 5.5. On considère les deux intégrales :

$$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx\,, \quad J = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx\,.$$

(1) A l'aide du changement de variables $u = \pi - x$, exprimer I en fonction de J.

(2) Calculer J et en déduire la valeur de I.

$u = \pi - x \quad du = -dx \quad \Rightarrow x = \pi - u$

$I = +\int_0^{\pi} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du = \int_0^{\pi} \frac{(\pi-u)\sin(u)}{1+\cos^2(u)}\,du = \pi J - \int_0^{\pi} \frac{u\sin(u)}{1+\cos^2(u)}\,du$

$J = \pi J - I \Rightarrow I = \frac{\pi J}{2}.$

$J = \int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}\,dx = +\int_{-1}^{1} \frac{du}{1+u^2} = \left[\arctan(u)\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} = J$

$I = \frac{\pi^2}{4}.$

$u = \cos(x)\,, \quad du = -\sin(x)\,dx$

Exercice 5.6. Calculer les primitives suivantes :

$$\int \frac{1}{x^2+2x-3}\,dx\,, \quad \int \frac{x}{x^2-2x-3}\,dx\,, \quad \int \frac{x+1}{x(x+2)}\,dx\,, \quad \int \frac{x+1}{x^2-x-6}\,dx\,,$$

$$\int \frac{x}{x^2-2x+2}dx\,, \quad \int \frac{x+1}{x^2+4}\,dx\,, \quad \int \frac{x+1}{x^2+x+1}\,dx\,, \quad \int \frac{1}{(x+1)(x^2+1)}\,dx.$$

1. $x^2 + 2x - 3 = (x+3)(x-1)$

$\int \frac{A}{x+3} + \frac{B}{x-1}\,dx = \frac{1}{4}\left[\ln(x-1) - \ln(x+3)\right].$

$$\frac{1\cancel{(x+3)}}{(x-1)\cancel{(x+3)}} = \frac{A\cancel{(x+3)}}{\cancel{x+3}} + \cancel{\frac{B\cancel{(x+3)}}{x-1}} \Rightarrow \frac{1}{x-1} = A$$

$$x=-3 \quad A \Rightarrow -\frac{1}{4}.$$

$$x=-3 \quad A = \frac{1}{-3-1} \Rightarrow A = -\frac{1}{}$$