TD34 : Maths.

**Exercice 5.11.** Calculer les intégrales suivantes :

$$I = \int_0^1 \sqrt{x^2+x+1}\,dx\,,\quad J = \int_1^3 \frac{1}{\sqrt{4x-x^2}}\,dx\,,\quad K = \int_0^{1/2} \sqrt{\frac{x}{1-x}}\,dx\,,\quad L = \int_1^2 x\sqrt{x^2-2x+5}\,dx.$$

*indication :* pour le calcul de $L$, on introduira l'unique $\alpha$ tel que $\mathrm{sh}(\alpha) = \frac{1}{2}$.

① $$\int_0^1 \sqrt{x^2+x+1}\,dx = \int_0^1 \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{\sqrt{3}}{2}\int_0^1 \sqrt{\left(\frac{(x+\frac{1}{2})}{\sqrt{3}/2}\right)^2+1} = \frac{\sqrt{3}}{2}\int_0^1 \sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx = \frac{3}{4}\int_{1/\sqrt{3}}^{\sqrt{3}} \sqrt{u^2+1}\,du$$

$$u = \left(\frac{2x+1}{\sqrt{3}}\right)^2 \quad du = \frac{2\,dx}{\sqrt{3}}$$

$$v = \mathrm{sh}(u) \quad dv = \cosh(u)du \quad \Rightarrow \frac{3}{4}\int_{\mathrm{arcsinh}(\frac{1}{\sqrt{3}})}^{\mathrm{arcsinh}(\sqrt{3})} \cosh^2(v)\,dv$$

$$\mathrm{arcsinh}(a) = \ln\left|a+\sqrt{a^2+1}\right|$$

$$\mathrm{arcsinh}\left(\frac{1}{\sqrt{3}}\right) = \ln\left|\frac{1}{\sqrt{3}}+\sqrt{\frac{1}{3}+\frac{3}{3}}\right| = \ln\left|\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\right| = \frac{1}{2}\ln(3).$$

$$\mathrm{arcsinh}(\sqrt{3}) = \ln(\sqrt{3}+2)$$

$$\Rightarrow \frac{3}{4}\int_{\frac{1}{2}\ln(3)}^{\ln(\sqrt{3}+2)} \cosh^2(v)\,dv \quad \Bigg\| \quad \cosh^2(x) = \frac{e^{2x}+e^{-2x}+2}{4} =$$

$$\Rightarrow \frac{3}{8}\int_{\frac{1}{2}\ln(3)}^{\ln(\sqrt{3}+2)} (\cosh(2x)+1)\,dx = \frac{3}{8}\int_{\ln\sqrt{3}}^{\ln(\sqrt{3}+2)} dx + \frac{3}{8}\int_{\ln\sqrt{3}}^{\ln(\sqrt{3}+2)} \cosh(2x)\,dx.$$

$$= \frac{3}{8}\left(\ln(\sqrt{3}+2)-\ln\sqrt{3}\right) + \frac{3}{8}\Big[2\sinh(2x)\Big]_{\ln\sqrt{3}}^{\ln(\sqrt{3}+2)}$$

$$= \frac{3}{8}\ln\left(1+\frac{2}{\sqrt{3}}\right) + \frac{3}{8}\left(\sinh(2\ln(\sqrt{3}+2)) - \sinh(2\ln(\sqrt{3}))\right) = \frac{3}{8}\left(\ln\left(1+\frac{2}{\sqrt{3}}\right) + (\sqrt{3}+2)^2 - (\sqrt{3}+2)^{-2} - 3 + \frac{1}{3}\right.$$

$$). \int_1^3 \frac{dx}{\sqrt{4x-x^2}} = \int_1^3 \frac{dx}{\sqrt{(x-2)^2-4}} = \frac{1}{2}\int_1^3 \frac{dx}{\sqrt{\left(\frac{x-2}{2}\right)^2-1}}$$

$$u = \frac{x-2}{2} \quad du = \frac{1}{2}dx. \Rightarrow \int_{-1/2}^{1/2} \frac{du}{\sqrt{u^2-1}} = \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{3}.$$

$$\left(\sqrt{x^2-x}\right)' = \frac{2x-1}{2\sqrt{x^2-x}}$$

$$K = \int_0^{1/2} \sqrt{\frac{x}{1-x}}\,dx = \int_0^{1/2} \sqrt{\frac{x^2}{x(1-x)}}\,dx = \int_0^{1/2} \frac{x}{\sqrt{x-x^2}} = \int_0^{1/2} \frac{x-\frac{1}{2}}{\sqrt{x^2-x}}dx + \frac{1}{2}\int_0^{1/2} \frac{dx}{\sqrt{x^2-x}}$$

$$u = x^2 - x. \quad du = (2x-1)dx$$

$$\int_0^{1/4} \frac{-\frac{1}{2}du}{\sqrt{u}} = -\frac{1}{4}\left[\sqrt{u}\right]_0^{1/4} = -\frac{1}{4}\left[\frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{2}}{8}$$

$$L = \int_1^2 x\sqrt{x^2-2x+5}\,dx = \int_1^2 (x-1)\sqrt{x^2-2x+5}\,dx + \int_1^2 \sqrt{x^2-2x+5}\,dx$$

$$\Rightarrow u = x^2-2x+5 \quad du = (2x-2)dx$$

$$\frac{1}{2}\int_4^5 \sqrt{u}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_4^5 = \frac{1}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}\left(5^{3/2}-8\right)$$

$$\int_1^2 \sqrt{x^2-2x+5}\,dx = \int_1^2 \sqrt{(x-1)^2+4}\,dx = 2\int_1^2 \sqrt{\left(\frac{x-1}{2}\right)^2+1}\,dx$$

$$u = \frac{x-1}{2}. \quad du = \frac{1}{2}dx. \Rightarrow 4\int_0^{1/2} \sqrt{u^2+1}\,du.$$

$$u = \sinh(v). \qquad du = \cosh(v)\,dv \qquad \Rightarrow 4\int_0^{\operatorname{arcsinh}(\frac{1}{2})} \cosh^2(v)\,dv = \dots$$