TD 38 : Maths.

Exercice 5.19. Soit $f : \mathbb{R} \longrightarrow \mathbb{R}$ une fonction T-périodique et continue. Montrer que pour tout $a \in \mathbb{R}$, on a
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

$$\int_a^{a+T} f(x)dx = \int_a^0 f(x)dx + \int_0^T f(x)dx + \int_T^{a+T} f(t)dt.$$

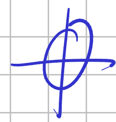
$$= \int_0^T f(x)dx + \int_T^{a+T} f(x)dx - \int_0^a f(x)dx.$$

$\Rightarrow u = x+T.\ dx = du. \quad \Rightarrow \int_0^a f(x)dx = \int_T^{a+T} f(u-T)du = \int_T^{a+T} f(u)du.$

Exercice 5.20 (Sommes de Riemman). Calculer les limites des suites définies par :

(1) $u_n = \frac{1}{n}\left(\sin(\pi/n) + \sin(2\pi/n) + \cdots + \sin(n\pi/n)\right)$;

(2) $v_n = n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n^2)}\right)$.



1. $u_n = \frac{1}{\pi}\left(\frac{2\pi-\pi}{n}\sum_{k=1}^{n}\sin\left(0+k\frac{2\pi-\pi}{n}\right)\right) = \frac{1}{\pi}\int_\pi^{2\pi}\sin(t)dt = -\frac{1}{\pi}\left[\cos(x)\right]_\pi^{2\pi} = \frac{2}{\pi}.$

$f(x) = x^{-2}$

$v_n = n\sum_{k=1}^{n}\frac{1}{(n+k)^2} = n\sum_{k=1}^{n} f(n+k) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{n+k}{n}\right) = \frac{2-1}{n}\sum_{k=1}^{n} f\left(1+\frac{k}{n}\right) \xrightarrow[+\infty]{} \int_1^2 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2}+1 = \frac{1}{2}.$

(1) Soit $f : [a,b] \to \mathbb{R}$ continue et $g : \mathbb{R} \to \mathbb{R}$ convexe. Démontrer que
$$g\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b g(f(t))\,dt.$$

Indication : on pourra considérer les sommes de Riemann $u_n = \frac{1}{b-a}\sum_{k=0}^{n-1} f\left(a + k\frac{b-a}{n}\right)$.

(2) En déduire que
$$\exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \exp(f(t))\,dt.$$

f convexe : $\forall \lambda \in [0,1]$, $f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y)$.

1. $g\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)$. On pose $u_n = \frac{1}{b-a}\sum_{k=0}^{n-1} f\left(a+k\frac{b-a}{n}\right)$

Soient $x_1 < x_2 < x_3 \dots < x_n$ et $t_1, t_2, t_3 \in [0,1]$ tq $\sum t_i = 1$.

On a $g(t_1x_1 + t_2x_2 + \dots + t_nx_n) \leq t_1g(x_1) + t_2g(x_2) + \dots + t_n(g_{x_n})$.

$$= \frac{b-a}{n}\sum_{k=0}^{n-1}\underbrace{f\left(a+k\frac{b-a}{n}\right)}_{x_k} = t_k\sum_{k=0}^{n-1}x_k$$

$$\frac{1}{b-a}\left(\frac{b-a}{n}\right) = \frac{\cancel{b-a}}{(\cancel{b-a})n} = \frac{1}{n}\sum_{k=0}^{n-1}x_k.$$

$$g\left(\frac{1}{n}\sum_{k=0}^{n-1}x_k\right) \leq \frac{1}{n}g(x_0) + \dots + \frac{1}{n}g(x_n) = \frac{1}{n}\left(\sum_{k=0}^{n}g(x_k)\right)$$

$$= \frac{1}{n}\left(\sum_{k=0}^{n}g\left(f\left(a+k\frac{b-a}{n}\right)\right)\right)$$

on a $= g\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(a+k\frac{b-a}{n}\right)\right)$ $\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}g\left(f\left(a+k\frac{b-a}{n}\right)\right) = \frac{1}{b-a}\int_a^b g(f(t))\,dt$.

ok.

et donc on a $\frac{1}{b-a}$

$= g\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)$.